



On the Number of Conjugacy Classes of Subgroups of a Finite Group

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Abstract. Let $k'(G)$ and $L(G)$ be the number of conjugacy classes of subgroups and the subgroup lattice of a finite group G , respectively. Our objective is to study some aspects related to the ratios $d'(G) = \frac{k'(G)}{|L(G)|}$ and $d^*(G) = \min\{d'(S) \mid S \text{ is a section of } G\}$ which measure how close G is to being a Dedekind group. We prove that the set containing the values $d'(G)$, as G ranges over the class of nilpotent groups, is dense in $[0, 1]$. A nilpotency criterion is obtained by proving that if $d^*(G) > \frac{2}{3}$, then G is nilpotent and information on its structure is given. We also show that if $d^*(G) > \frac{4}{5}$, then G is an Iwasawa group. Finally, we deduce a result which ensures that a p -group of order p^n ($n \geq 3$) is a Dedekind group. This last result can be extended to the class of nilpotent groups and it also highlights the second maximum value of d^* on the class of p -groups of order p^n .

Mathematics Subject Classification. Primary 20D60; Secondary 20E45, 20D15.

Keywords. Conjugacy classes of subgroups, Dedekind group, section of a group, p -group.

1. Introduction

All groups considered in this paper are finite and $n \geq 1$ denotes an integer. Let G be a group and p be a prime number. We denote by $k(G)$, $k'(G)$, $\nu(G)$, $N(G)$ and $L(G)$ the number of conjugacy classes, the number of conjugacy classes of subgroups, the number of conjugacy classes of non-normal subgroups, the normal subgroup lattice, and the subgroup lattice of G , respectively. The cyclic group of order n is denoted by C_n , the dihedral group D_{2n} ($n \geq 3$) has the following structure

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, yx = x^{n-1}y \rangle,$$

M_{p^n} ($n \geq 4$ if $p = 2$; $n \geq 3$ if p is odd) is a modular p -group (i.e., a p -group whose subgroup lattice is modular) given by

$$M_{p^n} = \langle x, y \mid x^{p^{n-1}} = y^p = 1, yx = x^{p^{n-2}+1}y \rangle,$$

while He_p is the Heisenberg group modulo p (p is odd) and its representation is

$$He_p = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle.$$

We also recall that an Iwasawa group is a modular nilpotent group.

A way to measure how close G is to being abelian is given by its commutativity degree

$$d(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

Theorem IV of [6] states that the numerator is equal to $k(G)|G|$, so the commutativity degree of G can be also expressed as

$$d(G) = \frac{k(G)}{|G|}.$$

In this paper, we study another ratio that is defined similarly, the main difference being that we switch from working with the elements of G to operating with its subgroups. Hence, we are going to investigate the ratio

$$d'(G) = \frac{k'(G)}{|L(G)|}.$$

This quantity measures how close is G from being a Dedekind group (i.e., all its subgroups are normal). It is clear that all abelian groups are Dedekind, while the structure of the non-abelian Dedekind groups, which are called Hamiltonian groups, is also known (see statement 5.3.7 of [12]).

Some properties of d' are straightforward. For any group G , we have $0 < d'(G) \leq 1$, and the equality $d'(G) = 1$ holds if and only if G is a Dedekind group. If G_1 and G_2 are two groups such that $G_1 \cong G_2$, then $d'(G_1) = d'(G_2)$. The converse does not hold even for non-Dedekind groups. For instance, if $G_1 \cong C_3 \rtimes C_8$ (SmallGroup(24,1)) and $G_2 \cong C_3 \times D_8$ (SmallGroup(24,10)), we have $d'(G_1) = d'(G_2) = \frac{4}{5}$ and $G_1 \not\cong G_2$. Finally, if I is a finite set with $|I| \geq 2$ and $(G_i)_{i \in I}$ is a family of groups of coprime orders, then

$$d'\left(\times_{i \in I} G_i\right) = \prod_{i \in I} d'(G_i),$$

so d' is multiplicative.

Our main results are listed below and their proofs are included in the following two sections. In Sect. 2, we explicitly compute $d'(G)$ with G belonging to specific classes of groups. Given an integer $a \geq 1$, we show that all ratios of the form $\frac{a}{a+1}$ are attained by d' . The main result of this section is the following one:

Theorem A. *Let $\mathcal{N}$ be the class of nilpotent groups. Then, the set*

$$\{d'(G) \mid G \in \mathcal{N}\}$$

is dense in $[0, 1]$.

In Sect. 3, we justify that there is no constant $c \in (0, 1)$ such that if $d'(G) > c$, then G is a nilpotent/Iwasawa/Dedekind group. However, a nilpotency criterion and a criterion for G to be an Iwasawa group can be obtained by replacing $d'(G)$ with $d^*(G)$, where

$$d^*(G) = \min\{d'(S) \mid S \text{ is a section of } G\}.$$

More exactly, we justify that the following results hold:

Theorem B. *Let G be a group. If $d^*(G) > \frac{2}{3} = d^*(S_3) = d'(S_3)$, then G is a nilpotent group. Moreover, if $2 \nmid |G|$, then G is the direct product of some modular p -groups.*

Theorem C. *Let G be a group. If $d^*(G) > \frac{4}{5} = d^*(D_8) = d'(D_8)$, then G is an Iwasawa group.*

By using the classification of minimal non-Dedekind p -groups, we also determine a criterion which guarantees that p -groups of fixed orders are Dedekind groups. This result can be extended to the class of nilpotent groups and it is stated below:

Theorem D. *Let $n \geq 3$ be an integer and G be a p -group of order p^n .*

- i) Assume that $p = 2$ and $n = 3$. If $d^*(G) > \frac{4}{5} = d^*(D_8) = d'(D_8)$, then G is a Dedekind group.*
- ii) Assume that $n \geq 4$ if $p = 2$, and $n \geq 3$ if p is odd. If $d^*(G) > \frac{(n-2)(p+1)+4}{(n-1)(p+1)+2} = d^*(M_{p^n}) = d'(M_{p^n})$, then G is a Dedekind group.*

The paper concludes with a list of open problems.

2. On the Values of d'

As a preliminary result, we recall the main theorem of [3]. It outlines the classification of groups having one conjugacy class of non-normal subgroups. These groups are close to being Dedekind groups and this aspect will be helpful in obtaining some of our results.

Lemma 2.1. *Let G be a group. Then, $\nu(G) = 1$ if and only if G is isomorphic to one of the following groups:*

- i) M_{p^n} .*
- ii) A non-abelian split extension $G_{p,q,n} = N \rtimes P$ where $N \cong C_p$, $P \cong C_{q^{n-1}}$, $[N, \Phi(P)] = 1$, p, q are primes, such that $q \mid p-1$ and $n \geq 2$.*

In what follows we obtain explicit formulas for computing $d'(G)$ when G is a group with $\nu(G) = 1$, a dihedral 2-group or a Heisenberg group modulo p .

Proposition 2.2. *The following results hold:*

- i) $d'(M_{p^n}) = \frac{(n-2)(p+1)+4}{(n-1)(p+1)+2}$;
- ii) $d'(G_{p,q,n}) = \frac{2n}{2n+p-1}$;
- iii) $d'(D_{2^n}) = \frac{3n-1}{2^n+n-1}$;
- iv) $d'(He_p) = \frac{2p+5}{p^2+2p+4}$.

Moreover, the sequences $(d'(M_{p^n}))_{n \geq 3}$ (p is odd), $(d'(M_{2^n}))_{n \geq 4}$, $(d'(G_{p,q,n}))_{n \geq 2}$ are strictly increasing, while the sequences $(d'(D_{2^n}))_{n \geq 3}$, $(d'(He_p))_{p \geq 3}$ are strictly decreasing.

Proof. Note that $k'(G) = |N(G)| + \nu(G)$ for any group G . According to Lemma 3.3 of [15], we have

$$|L(M_{p^n})| = (n-1)(p+1) + 2 \text{ and } |L(D_{2^n})| = 2^n + n - 1.$$

i) As stated in Lemma 3.5 of [15], we have

$$|N(M_{p^n})| = (n-2)(p+1) + 3.$$

Since $\nu(M_{p^n}) = 1$, the conclusion follows.

ii) The subgroup $P \cong C_{q^{n-1}}$ is the representative of the unique conjugacy class of non-normal subgroups of $G_{p,q,n}$. This class contains p conjugates. Besides N and its improper subgroups, $G_{p,q,n}$ has one normal subgroup of order pq^i and one normal subgroup of order q^i for each $i \in \{1, 2, \dots, n-2\}$. Therefore,

$$k'(G_{p,q,n}) = 2n \text{ and } |L(G_{p,q,n})| = 2n + p - 1,$$

which leads to the desired result.

iii) Theorem 3.3 of [5] outlines the conjugacy classes of subgroups of any dihedral group. According to this result, D_{2^n} has 3 conjugacy classes of subgroups of order 2^i for each $i \in \{1, 2, \dots, n-1\}$. Hence,

$$k'(D_{2^n}) = 3n - 1,$$

and the conclusion follows.

iv) The proper non-trivial normal subgroups of He_p are $Z(He_p) \cong C_p$ and its $p+1$ maximal subgroups isomorphic to C_p^2 . There are also $p+1$ non-trivial conjugacy classes of subgroups, each being formed of p conjugates isomorphic to C_p . We conclude that

$$k'(He_p) = 2p + 5 \text{ and } |L(He_p)| = p^2 + 2p + 4,$$

so we easily obtain the value of $d'(He_p)$.

Regarding the second part of this proposition, we only justify the monotonicity of the first sequence. Let p be a fixed odd prime and $f : [3, \infty) \rightarrow \mathbb{R}$ be a function given by

$$f(x) = \frac{(x-2)(p+1)+4}{(x-1)(p+1)+2}, \quad \forall x \in [3, \infty).$$

We have

$$f'(x) = \frac{p^2 - 1}{[(x-1)(p+1) + 2]^2} > 0, \forall x \in (3, \infty),$$

so f is strictly increasing on $[3, \infty)$. Hence, the sequence $(f(n))_{n \geq 3} = (d'(M_{p^n}))_{n \geq 3}$ is strictly increasing. $\blacksquare$

The following result is a consequence of our previous proposition. It shows that $d'(G)$ can be made arbitrarily large or small as $|G|$ increases.

Corollary 2.3. *The following results hold:*

$$\lim_{n \rightarrow \infty} d'(M_{p^n}) = \lim_{n \rightarrow \infty} d'(G_{p,q,n}) = 1 \text{ and } \lim_{n \rightarrow \infty} d'(D_{2^n}) = \lim_{p \rightarrow \infty} d'(He_p) = 0.$$

It is clear that the values of d' belong to $S = [0, 1] \cap \mathbb{Q}$. We proceed our study by showing that specific values from S are attained by d' . Then, in the proof of Theorem A, we justify that each value of S is an adherent point of the set $\{d'(G) \mid G \in \mathcal{N}\}$, where $\mathcal{N}$ is the class of nilpotent groups. The classification given by Lemma 2.1 and the numerical results outlined in Proposition 2.2 play a significant role for both purposes.

Corollary 2.4. *Let $a \geq 1$ be an integer. Then, there is a group G , such that $d'(G) = \frac{a}{a+1}$.*

Proof. If $a = 1$, we take $G = G_{5,2,2} \cong D_{10}$ and we have $d'(G) = \frac{1}{2}$. Suppose that $a \geq 2$. Then, we can choose $G = G_{3,2,a}$ and we get $d'(G) = \frac{a}{a+1}$. $\blacksquare$

Proof of Theorem A. Let $x \in S$. We aim to show that there is a sequence of groups $(G_n)_{n \geq n_0} \subseteq \mathcal{N}$, such that

$$\lim_{n \rightarrow \infty} d'(G_n) = x.$$

If $x = 0$ or $x = 1$, we are done by Corollary 2.3. Suppose that $x = \frac{a}{b}$ where $1 \leq a < b$. Also, let $(p_k)_{k \geq 1}$ be the strictly increasing sequence of odd prime numbers. We construct $b - a$ strictly increasing and disjoint subsequences $(p_{1_n})_{n \geq 1}, (p_{2_n})_{n \geq 1}, \dots, (p_{(b-a)_n})_{n \geq 1}$ of $(p_k)_{k \geq 1}$. Also, for each $i \in \{1, 2, \dots, b - a\}$, consider the sequence of groups $(G_{i_n})_{n \geq 1}$ where

$$G_{i_n} = M_{p_{i_n}^{a+i+1}}.$$

Note that each of these $b - a$ sequences of groups is well defined, since we work only with odd prime numbers and $a + i + 1 \geq 3$. Finally, we define the sequence $(G_n)_{n \geq 1} \subseteq \mathcal{N}$ whose general term is

$$G_n = \times_{i=1}^{b-a} G_{i_n}.$$

The factors of the above direct product are of coprime orders, since the sequences $(p_{i_n})_{n \geq 1}$ are disjoint. Hence, we are able to apply the multiplicativity

property of d' as follows:

$$\begin{aligned} d'(G_n) &= d'\left(\times_{i=1}^{b-a} G_{i_n}\right) = \prod_{i=1}^{b-a} d'(G_{i_n}) = \prod_{i=1}^{b-a} d'(M_{p_{i_n}^{a+i+1}}) \\ &= \prod_{i=1}^{b-a} \frac{(a+i-1)(p_{i_n}+1)+4}{(a+i)(p_{i_n}+1)+2}. \end{aligned}$$

As n tends to infinity, we obtain

$$\lim_{n \rightarrow \infty} d'(G_n) = \prod_{i=1}^{b-a} \lim_{n \rightarrow \infty} \frac{(a+i-1)(p_{i_n}+1)+4}{(a+i)(p_{i_n}+1)+2} = \prod_{i=1}^{b-a} \frac{a+i-1}{a+i} = \frac{a}{b} = x.$$

Therefore, we conclude that each value of S is an adherent point of the set $\{d'(G) \mid G \in \mathcal{N}\}$. This implies that

$$S \subseteq \overline{\{d'(G) \mid G \in \mathcal{N}\}},$$

so

$$[0, 1] = \overline{S} \subseteq \overline{\{d'(G) \mid G \in \mathcal{N}\}}.$$

Since the reverse inclusion is obvious, the proof is complete. $\blacksquare$

Theorem A can be easily extended to any class of groups containing $\mathcal{N}$ as follows.

Corollary 2.5. *Let $\mathcal{C}$ be a class of groups, such that $\mathcal{N} \subseteq \mathcal{C}$. Then, $\{d'(G) \mid G \in \mathcal{C}\}$ is dense in $[0, 1]$.*

3. Criteria Ensuring that a Finite Group Is Nilpotent/Iwasawa/Dedekind

Let G be a group. Given a group theoretic property P , G is said to be a minimal non- P group if G does not satisfy P and all its proper subgroups do. The following lemma recalls the structure and some properties of the minimal non-nilpotent groups. These are also known as Schmidt groups. The listed results are outlined in [13] and [11].

Lemma 3.1. *Let G be a Schmidt group. Then, the following statements hold:*

- i) $G = P \rtimes Q$ where P is a normal Sylow p -subgroup, Q is a cyclic Sylow q -subgroup of G and p, q are distinct primes.
- ii) $Z(G) = \Phi(G) = \Phi(P) \times \Phi(Q)$.
- iii) $\frac{P}{\Phi(P)} \cong C_p^r$, where r is the multiplicative order of p modulo q .
- iv) If N is a proper normal subgroup of G , then N does not contain Q and either $P \subseteq N$ or $N \subseteq Z(G)$.

Let $c \in (0, 1)$ and G be a group. A condition such as $d'(G) > c$ cannot guarantee that G is a nilpotent/Dedekind group. Indeed, according to Corollary 2.3 for a sufficiently large value of n , say n_c , we would get $d'(M_{p^{n_c}}) > c$

and $d'(G_{p,q,n_c}) > c$, but $M_{p^{n_c}}$ is a non-Dedekind group, while G_{p,q,n_c} is non-nilpotent. Theorem B highlights a nilpotency criterion that is obtained by replacing the condition $d'(G) > c$ with $d^*(G) > c$, where

$$d^*(G) = \min\{d'(S) \mid S \text{ is a section of } G\}.$$

A useful property of d^* is that it behaves like a decreasing function defined on $L(G)$, i.e., if $H, K \in L(G)$ and $H \subseteq K$, then $d^*(H) \geq d^*(K)$. Also, d^* is multiplicative. Indeed, if I is a finite set with $|I| \geq 2$ and $(G_i)_{i \in I}$ is a family of groups of coprime orders, then all sections of $\prod_{i \in I} G_i$ are of the form $\prod_{i \in I} S_i$, where S_i is a section of G_i for any $i \in I$. Then,

$$d^*\left(\prod_{i \in I} G_i\right) = \prod_{i \in I} d^*(G_i).$$

Before proving Theorem B, we outline and justify some modularity criteria for p -groups.

Proposition 3.2. *Let G be a p -group.*

- i) If $d^*(G) > \frac{4}{5} = d^*(D_8) = d'(D_8)$, then G is a modular p -group.*
- ii) If p is odd and $d^*(G) > \frac{11}{19} = d^*(He_3) = d'(He_3)$, then G is a modular p -group.*

Proof. *i)* Suppose that G is a non-modular p -group. According to Lemma 2.3.3 of [14], G has a section S isomorphic to:

- D_8 , if $p = 2$;
- He_p , if p is odd.

We also know that $d^*(G) \leq d'(S)$. We use the results stated in Proposition 2.2. If $p = 2$, we get

$$d^*(G) \leq \frac{4}{5} = d'(D_8),$$

a contradiction. If p is odd, then

$$d^*(G) \leq d'(He_p) \leq \frac{11}{19} = d'(He_3) < \frac{4}{5},$$

a contradiction.

The above reasoning also justifies statement *ii)*, so the proof is complete. ■

We proceed by proving Theorem B.

Proof of Theorem B. Suppose that G is a counterexample of minimal order. Then, G is non-nilpotent. Let $H \in L(G)$, such that $H \neq G$. Then,

$$d^*(H) \geq d^*(G) > \frac{2}{3},$$

so H is nilpotent. Consequently, G is a Schmidt group.

We consider the section $S = \frac{G}{Z(G)}$ of G . By using the information introduced in Lemma 3.1, we deduce that $|S| = p^r q$ and, by the correspondence theorem, S has a normal subgroup

$$H = \frac{P \times \Phi(Q)}{Z(G)} \cong C_p^r.$$

On the other hand, a subgroup $K \cong C_q$ of S cannot be normal. Indeed, if $K \in N(S)$, then G would have a proper normal subgroup N with $|N| = q \cdot |Z(G)| = |\Phi(P)| \cdot |Q|$. By Lemma 3.1 *iv*), we would also have that $|P|$ divides $|N|$ and this leads to a contradiction. Hence, we conclude that

$$S = H \rtimes K \cong C_p^r \rtimes C_q$$

and K acts faithfully on H . Moreover, all proper subgroups of S are nilpotent, so S is also a Schmidt group. By Lemma 3.1 *ii*), we have $Z(S) = \{1\}$.

Note that S cannot have a subgroup L_i of order $p^i q$ with $1 \leq i < r$. If such a subgroup would exist, then it would be nilpotent, so we can assume that there is a subgroup $H_i \cong C_p^i$ of H , such that $L_i = H_i \times K \cong C_p^i \times C_q$. Then, K centralizes H_i . Since H also centralizes H_i , it follows that $S \subseteq C_S(H_i)$. Then, $H_i \subseteq Z(S) = \{1\}$, a contradiction.

We denote $|L(C_p^r)|$ by $a_{p,r}$. Taking into account that S has p^r conjugate subgroups isomorphic to C_q and there are no subgroups of order $p^i q$ with $1 \leq i < r$, we obtain

$$|L(S)| = a_{p,r} + p^r + 1.$$

Let $i \in \{1, 2, \dots, r-1\}$ and let H_i be a subgroup of S with $|H_i| = p^i$. By our previous discussion on the non-existence of subgroups of order $p^i q$, we deduce that H_i is not normal in S . Also, H is an abelian maximal subgroup of S and $H_i \subset H$, so the size of the conjugacy class of H_i is

$$[S : N_S(H_i)] = [S : H] = q.$$

It is known that the number of subgroups of S that are isomorphic to H_i is given by the Gaussian coefficient

$$\begin{bmatrix} r \\ i \end{bmatrix}_p = \frac{(p^r - 1)(p^{r-1} - 1) \cdots (p^{r-i+1} - 1)}{(p^i - 1)(p^{i-1} - 1) \cdots (p - 1)}$$

(see, for example, Exercise 1.74 of [1]). Hence, these subgroups are partitioned into $\frac{1}{q} \begin{bmatrix} r \\ i \end{bmatrix}_p$ conjugacy classes. Besides them, S has 3 trivial conjugacy classes of subgroups whose representatives are 1, H , and S , respectively, and the conjugacy class of K . Therefore,

$$k'(S) = \frac{1}{q} \sum_{i=1}^{r-1} \begin{bmatrix} r \\ i \end{bmatrix}_p + 4 = \frac{a_{p,r} + 4q - 2}{q}.$$

Consequently, we have

$$d'(S) = \frac{a_{p,r} + 4q - 2}{q(a_{p,r} + p^r + 1)}.$$

The following step is to show that $d'(S) \leq \frac{2}{3}$. This would contradict that $d^*(G) > \frac{2}{3}$ and the first part of our proof would be complete. Note that

$$d'(S) \leq \frac{2}{3} \iff (2q-3)a_{p,r} + 2q(p^r-5) + 6 \geq 0.$$

Since $q \geq 2$, in most cases, it suffices to justify that

$$p^r - 5 \geq 0. \quad (1)$$

This is true for $p \geq 5$, so there are two cases left to discuss: $p = 2$ and $p = 3$. We recall that $q \mid p^r - 1$. If we assume that $p = 2$, then $r \geq 2$. If $r \geq 3$, then (1) holds, while if $r = 2$, then $q = 3$ and $S \cong C_2^2 \rtimes C_3 \cong A_4$, so $d'(S) = \frac{1}{2} < \frac{2}{3}$. Finally, suppose that $p = 3$. If $r \geq 2$, then (1) holds, while if $r = 1$, then $q = 2$ and $S \cong C_3 \rtimes C_2 \cong S_3$, so $d'(S) = \frac{2}{3}$. Therefore, our initial assumption is false, so G is a nilpotent group.

We now prove the second part of the statement of Theorem B. Since G is nilpotent and $2 \nmid |G|$, it follows that $G \cong \times_{i=1}^k G_i$, where $G_1, G_2, \dots, G_k$ are the Sylow subgroups of G , each of them being of odd order. We have

$$d^*(G_i) \geq d^*(G) > \frac{2}{3} > \frac{11}{19}, \quad \forall i \in \{1, 2, \dots, k\}.$$

Then, each G_i is a modular group due to Proposition 3.2 ii). Hence, G is a direct product of modular p -groups. ■

Regarding the second part of the statement of Theorem B, we mention that the structure of modular p -groups is determined by Iwasawa in [9] (see also Theorem 2.3.1 of [14]). More exactly, P is a modular p -group if and only if one of the following holds:

- $P \cong Q_8 \times C_2^n$ ($n \geq 0$);
- P contains an abelian normal subgroup A , such that $\frac{P}{A}$ is cyclic; in addition, there is $g \in P$ and a positive integer n , such that $P = A\langle g \rangle$ and $ga = a^{p^n+1}g$ for all $a \in A$, with $n \geq 2$ if $p = 2$.

Also, according to Exercise 3 from Section 2.4 of [14], the fact that G is a direct product of modular p -groups is equivalent to any of the following:

- G is an Iwasawa group;
- all subgroups of G are permutable.

These additional details are useful for proving Theorem C.

Proof of Theorem C. Since

$$d^*(G) > \frac{4}{5} > \frac{2}{3},$$

G is a nilpotent group according to Theorem B. Then, G is the direct product of its Sylow subgroups. Let P be one of them. We have

$$d^*(P) \geq d^*(G) > \frac{4}{5},$$

so P is a modular p -group by Proposition 3.2 i). Consequently, G is an Iwasawa group, as desired. ■

We mention that the lower bounds that appear in the statements of Proposition 3.2 and Theorems B and C are the best possible ones, since D_8 and He_3 are non-modular p -groups, while S_3 is a non-nilpotent group.

A classification of minimal non-Dedekind groups is obtained in [4] (see also Lemma 3.1 of [8]). The following result illustrates the structure of minimal non-Dedekind p -groups only. The first two types of groups are also minimal non-abelian p -groups by Exercise 8a from Section 1 of [2]. The structure of their centers is described in Lemma 2.2 of [7] and is also recalled below.

Lemma 3.3. *Let G be a minimal non-Dedekind p -group. Then, G is isomorphic to one of the following groups:*

- i) $H_{p,s,t} = \langle x, y, z \mid x^{p^s} = y^{p^t} = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle$, where $s \geq t \geq 1$ and, if $p = 2$, then $n = s + t \geq 3$; moreover, $Z(H_{p,s,t}) = \langle x^p \rangle \times \langle y^p \rangle \times \langle z \rangle$;
- ii) $K_{p,s,t} = \langle x, y \mid x^{p^s} = y^{p^t} = 1, yx = x^{p^{s-1}+1}y \rangle$, where $s \geq 2, t \geq 1$ and, if $p = 2$, then $n = s + t \geq 4$; moreover $Z(K_{p,s,t}) = \langle x^p \rangle \times \langle y^p \rangle$;
- iii) Q_{16} .

A group G is a q -self-dual group if every quotient of G is isomorphic to a subgroup of G . Theorems 3 and 4 of [10] list the p -groups G all of whose subgroups are q -self-dual. For odd primes, the obtained classification is valid under the assumption that $\Omega_1(G)$ is abelian. The following preliminary result is a consequence of these theorems.

Lemma 3.4. M_{p^n} is a q -self-dual group.

The following result shows that the values of d' and d^* are equal for some p -groups.

Proposition 3.5. *The following equalities hold:*

- i) $d'(M_{p^n}) = d^*(M_{p^n})$;
- ii) $d'(He_p) = d^*(He_p)$.

Proof. i) Let $G = M_{p^n} \cong K_{p,n-1,1}$. Then, G is a minimal non-Dedekind group by Lemma 3.3. Let $H, K \in L(G)$, such that $K \in N(H)$ and $H \neq G$. Then, H is a Dedekind group and this property is inherited by the section $\frac{H}{K}$. Hence, we deduce that

$$d^*(G) = \min \left\{ d' \left(\frac{G}{N} \right) \mid N \in N(G) \right\}.$$

Since G is a q -self-dual group and all its proper subgroups are abelian, $\frac{G}{1} \cong G$ is its only non-Dedekind section. Therefore, $d^*(G) = d'(G)$.

ii) The only non-Dedekind section of $G \cong He_p$ is isomorphic to G , so the conclusion follows. ■

Let $c \in (0, 1)$. The idea of replacing the hypothesis $d'(G) > c$ with $d^*(G) > c$ in order to obtain a criterion for G to be a Dedekind group does not work. Indeed, due to Proposition 3.5, the two conditions are equivalent for $G \cong M_{p^n}$ and we explained that there is a non-Dedekind group $M_{p^{n_c}}$, such that $d'(M_{p^{n_c}}) = d^*(M_{p^{n_c}}) > c$. However, we are able to show that such a criterion can be obtained if we work with p -groups of fixed orders. Theorem D illustrates this result and our next objective is to prove it. We justify a preliminary result first.

Lemma 3.6. *Let p be a prime number. The following results hold:*

- i) *If $p = 2$ and $n = s + t \geq 3$, then $H_{2,s,t}$ has a section isomorphic to D_8 .*
- ii) *If p is odd and $n = s + t \geq 2$, then $H_{p,s,t}$ has a section isomorphic to He_p .*
- iii) *If $p = 2, n = s + t \geq 5$ and $K_{2,s,t} \not\cong M_{2^n}$, then $K_{2,s,t}$ has a section isomorphic to M_{2^k} , where $k < n$.*
- iv) *If p is odd, $n = s + t \geq 4$ and $K_{p,s,t} \not\cong M_{p^n}$, then $K_{p,s,t}$ has a section isomorphic to M_{p^k} , where $k < n$.*

Proof. By Lemma 3.3, we know that

$$N = \langle x^p \rangle \times \langle y^p \rangle \cong C_{p^{s-1}} \times C_{p^{t-1}} \text{ and } M = \langle y^{p^{t-1}} \rangle \cong C_p$$

are central subgroups of $H_{p,s,t}$ and $K_{p,s,t}$, respectively. Hence, they are also normal subgroups.

i) & ii) Let $G = H_{p,s,t}$, $\bar{x} = xN$, $\bar{y} = yN$ and $\bar{z} = zN$. Then,

$$S = \frac{G}{N} = \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^p = \bar{y}^p = \bar{z}^p = N, [\bar{x}, \bar{z}] = [\bar{y}, \bar{z}] = N, [\bar{x}, \bar{y}] = \bar{z} \rangle$$

is a non-abelian section of order p^3 of G . If $p = 2$, it is easy to check that S has 5 elements of order 2, so $S \cong D_8$. Otherwise, p is odd and $S \cong He_p$.

iii) We proceed by induction on n . For the initial step, if $n = 5$, then one can use GAP [16]¹ to check that $K_{2,3,2}$ (SmallGroup(32, 4)) and $K_{2,2,3}$ (SmallGroup(32, 12)) have a section isomorphic to M_{16} .

Let $n \geq 6$ and suppose that the statement holds for any m , such that $5 \leq m < n$. Let $G = K_{2,s,t} \not\cong M_{2^n}$. Then, $t \geq 2$. We have

$$\frac{G}{M} \cong K_{2,s,t-1}.$$

If $K_{2,s,t-1} \cong M_{2^{n-1}}$, then we are done. Otherwise, by the inductive hypothesis, it follows that $\frac{G}{M}$ has a section

$$S = \frac{\frac{H}{M}}{\frac{K}{M}} \cong \frac{H}{K} \cong M_{2^k},$$

where $H, K \in L(G)$ and $K \in N(H)$. Then, G also has a section isomorphic to M_{2^k} and the proof is complete.

¹See Appendix A for the GAP code used to verify these results.

iv) This is also done by induction on n . If $n = 4$, then $G = K_{p,2,2}$ has a section

$$\frac{G}{M} \cong K_{p,2,1} \cong M_{p^3}.$$

Thus, the initial step is complete. The inductive step follows a similar argument as the one done in the proof of item iii), so we omit it. ■

We continue with proving Theorem D.

Proof of Theorem D. The result outlined in item i) is clear, since D_8 is the only non-Dedekind group of order 8.

Regarding item ii), let G be a p -group, such that

$$d^*(G) > d^*(M_{p^n}). \quad (2)$$

We proceed by induction on n . For the initial step, we work with groups of order 16 or p^3 , where p is odd. By checking the list of groups of order 16, we deduce that (2) holds if and only if G is a Dedekind group. In addition, the only non-Dedekind groups of order p^3 are M_{p^3} and He_p . By Propositions 2.2 and 3.5, we have

$$d^*(He_p) = d'(He_p) = \frac{2p+5}{p^2+2p+4} < \frac{p+5}{2p+4} = d'(M_{p^3}) = d^*(M_{p^3}).$$

Therefore, (2) holds again if and only if G is a Dedekind group.

Let $n \geq 5$ when $p = 2$, and $n \geq 4$ when p is odd. Assume that the statement holds for any m , such that $m < n$. If $p = 2$, we may assume that $m \geq 4$, whereas if p is odd, then $m \geq 3$. Let G be a p -group of order p^n such that (2) holds. Based on the parity of p , by using Propositions 2.2 and 3.5 again, we have

$$d^*(G) > d^*(M_{2^n}) \geq d^*(M_{32}) = \frac{13}{14} \text{ or } d^*(G) > d^*(M_{p^n}) \geq d^*(M_{p^4}) = \frac{2p+6}{3p+5}.$$

For the sake of contradiction, suppose that G is a non-Dedekind group. Let H be a maximal subgroup of G . By using Propositions 2.2 and 3.5, we have

$$d^*(H) \geq d^*(G) > d^*(M_{p^n}) > d^*(M_{p^{n-1}}).$$

Therefore, H is a Dedekind group by the inductive hypothesis. Consequently, G is a minimal non-Dedekind group. By Lemma 3.3, it follows that $G \cong H_{p,s,t}$ or $G \cong K_{p,s,t}$.

If $G \cong H_{p,s,t}$, then, according to Lemma 3.6, G has a section isomorphic to D_8 or He_p . In the first case, we get $d^*(G) \leq d'(D_8) = \frac{4}{5}$, contradicting $d^*(G) > \frac{13}{14}$. In the second case, we obtain $d^*(G) \leq d'(He_p) = \frac{2p+5}{p^2+2p+4}$, contradicting $d^*(G) > \frac{2p+6}{3p+5}$.

Suppose that $G \cong K_{p,s,t}$. By (2), we know that $G \not\cong M_{p^n}$. Hence, we can use Lemma 3.6 to conclude that G has a section isomorphic to M_{p^k} , where $k < n$. It follows that

$$d^*(G) \leq d^*(M_{p^k}),$$

but this contradicts (2) due to Propositions 2.2 and 3.5.

We conclude that our assumption is false, so G is a Dedekind group. $\blacksquare$

We end this section with an extension of Theorem D to the class of nilpotent groups. This result holds due to the multiplicativity property of d^* .

Corollary 3.7. *Let $p_1 < p_2 < \dots < p_k$ be prime numbers, G be a nilpotent group and $P_1, P_2, \dots, P_k$ be its Sylow subgroups, such that*

$$|P_i| = p_i^{n_i}, \text{ where } n_i \geq 3, \forall i \in \{1, 2, \dots, k\}.$$

i) Assume that $p_1 = 2$ and $n_1 = 3$. If $d^(P_1) > d^*(D_8)$ and*

$$d^*(P_i) > d^*(M_{p_i^{n_i}}), \forall i \in \{2, 3, \dots, k\},$$

then G is a Dedekind group.

ii) Assume that $n_1 \geq 4$ if $p_1 = 2$, and $n_1 \geq 3$ if p_1 is odd. If

$$d^*(P_i) > d^*(M_{p_i^{n_i}}), \forall i \in \{1, 2, \dots, k\},$$

then G is a Dedekind group.

The lower bounds that appear in the statements of Theorem D and Corollary 3.7 are the best possible ones, since D_8 and M_{p^n} are non-Dedekind groups. Note that Corollary 3.7 does not hold for non-nilpotent supersolvable groups. More exactly, regarding item *i*), we can take $G \cong C_{27} \rtimes Q_8$ (SmallGroup(216, 4)). Its Sylow subgroups are Dedekind groups, but G is non-Dedekind, since $d'(G) = \frac{2}{11}$. In what concerns item *ii*), we can choose $G_1 \cong C_3^3 \rtimes C_4^2$ (SmallGroup(432, 425)) and the Zassenhaus metacyclic group (see Theorem 11 of [17])

$$G_2 = \langle x, y \mid x^{6859} = y^{27} = 1, yx = x^{956}y \rangle \cong C_{6859} \rtimes C_{27}.$$

The Sylow subgroups of both groups are Dedekind. However, $d'(G_1) = \frac{89}{224}$ and G_2 has a non-normal subgroup $H \cong C_{27}$, so both groups are non-Dedekind.

4. Open Problems

We end our paper by enumerating some questions that can be further explored.

Problem 4.1. Let a, b be integers, such that $1 \leq a < b$. In Sect. 2, we proved that any ratio of the form $\frac{a}{a+1}$ can be achieved by computing d' for specific non-nilpotent supersolvable groups having one conjugacy class of non-normal subgroups. We also showed that each ratio of the form $\frac{a}{b}$ can be achieved as the limit of $(d'(G_n))_{n \geq 1}$ where $(G_n)_{n \geq 1}$ is a sequence of nilpotent groups formed as direct products of some modular p -groups. Hence, we pose the following question:

Is it true that there is a group G such that $d'(G) = \frac{a}{b}$?

To get an affirmative answer, it suffices to search for $b - a$ groups $G_1, G_2, \dots, G_{b-a}$ of coprime orders, such that

$$d'(G_i) = \frac{a + i - 1}{a + i}, \quad \forall i \in \{1, 2, \dots, b - a\}.$$

By taking $G = \prod_{i=1}^{b-a} G_i$ and making use of the multiplicativity of d' , we would get a positive answer.

Problem 4.2. Let G be a group with $d^*(G) > \frac{2}{3}$. According to Theorem B, it follows that G is a nilpotent group. Moreover, due to the same result, if $2 \nmid |G|$, then G is an Iwasawa group. Assume that $2 \mid |G|$. Then, G has a Sylow 2-subgroup P with

$$d^*(P) \geq d^*(G) > \frac{2}{3}.$$

What can be said about the structure of P ?

Note that if $\frac{2}{3}$ is replaced with $\frac{4}{5}$, an answer is given by Iwasawa's result that is recalled after proving Theorem B. One can find a 2-group P , such that $d^*(P) \in (\frac{2}{3}, \frac{4}{5})$. For instance, by checking the groups of order 16, then $P \cong C_2^2 \rtimes C_4$ (SmallGroup(16, 3)) and $d^*(P) = d'(P) = \frac{17}{23}$, or $P \cong C_2 \times D_8$ (SmallGroup(16, 11)) and $d^*(P) = d'(P) = \frac{27}{35}$.

Problem 4.3. Let $n \geq 3$ be an integer. Theorem D indicates the second maximum value attained by d^* on the class of p -groups of order p^n .

Which is the second maximum value achieved by d' on the same class of groups?

Assume that $p = 2$ and G is a non-Dedekind group of order 2^n . If $n = 3$, one can check that $d'(G) \leq \frac{4}{5} = d'(D_8)$, while if $n = 4$, we have $d'(G) \leq \frac{10}{11} = d'(M_{16})$. Also, for $n \in \{5, 6, 7\}$, GAP computations show that $d'(G) \leq d'(C_2^{n-5} \times (D_8 \circ Q_8))$. We mention that $C_2^k \times (D_8 \circ Q_8)$ is a generalized extraspecial group of order 2^n , where $n = 5 + k$ and $k \geq 0$.

Which are the minimum values achieved by d' and d^ on the same class of groups?*

If $p = 2$, it can be checked that $d^*(G) \geq d^*(D_{2^n})$ for any $n \in \{3, 4, 5, 6, 7\}$. Since any section of D_{2^n} is abelian or isomorphic to D_{2^k} where $k \leq n$, by using Proposition 2.2, we deduce that $d'(D_{2^n}) = d^*(D_{2^n})$. Hence, we also have that $d'(G) \geq d'(D_{2^n})$ for the same values of n .

Acknowledgements

The authors are grateful to the reviewer for his/her remarks which improved the initial version of the paper.

Author contributions M.S.L. and M.T. conducted the research. M.S.L. wrote the paper. Both authors reviewed the manuscript.

Funding The authors did not receive support from any organization for the submitted work.

Data Availability Statement No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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A. GAP Code

The following GAP code may be used to check most of our results.

*# A function that returns a list containing the
sections of a group G*

```
sections:=function(G)
  local u, H, N;
  u := [];
  for H in List(ConjugacyClassesSubgroups(G),
    Representative) do
    for N in NormalSubgroups(H) do
      Add(u, FactorGroup(H,N));
    od;
  od;
```

```

    return u;
end;

# A function that computes  $d'(G)$ 

dG:=function(G)
    local c;
    c := ConjugacyClassesSubgroups(G);
    return Length(c) / Sum(c, Size);
end;

# A function that computes  $d^*(G)$ 
# It should be applied to the output of sections(G),
# i.e. dStarG(sections(G))

dStarG:=function(l)
    return Minimum(List(l, dG));
end;

```

References

- [1] Aigner, M.: A Course in Enumeration. Springer, Berlin (2007)
- [2] Berkovich, Y.: Groups of Prime Power Order, vol. 1. Walter de Gruyter GmbH & Co. KG, Berlin (2008)
- [3] Brandl, R.: Groups with few non-normal subgroups. Commun. Algebra **23**(6), 2091–2098 (1995)
- [4] Chen, S.D.: Classification of finite groups whose maximal subgroups are Dedekind groups. J. Zhejiang Univ. Sci. Ed. **29**(2), 121–124 (2002)
- [5] Conrad, K.: Dihedral groups II, <https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf>
- [6] Erdős, P., Turán, P.: On some problems of a statistical group-theory. IV. Acta. Math. Acad. Sci. Hungar **19**, 413–435 (1968)
- [7] Fouladi, S., Orfi, R.: The automorphism group of a finite minimal non-abelian p-group. Math. Rep. (Bucur.) **16**(66), 133–139 (2014)
- [8] Han, Z., Chen, G., Shi, H.: On minimal non-NSN-groups. J. Korean Math. Soc. **50**(3), 579–589 (2013)
- [9] Iwasawa, K.: Über die endlichen Gruppen und die Verbände ihrer Untergruppen. J. Fac. Imp. Univ. Tokyo Sect. **I**(4), 171–199 (1941)
- [10] Janko, Z.: Finite nonabelian p-groups all of whose subgroups are q-self dual. J. Algebra Appl. **13**(6), 1450008 (2014)
- [11] Monakhov, V.S.: Schmidt subgroups, its existence, and some of their applications, Vol. Section 1, Tr. Ukraini. Mat. Congr. 2001, Kiev, 2002, 81–90
- [12] Robinson, D.J.S.: A Course in the Theory of Groups, 2nd edn. Springer, New York (1996)

- [13] Schmidt, O.Y.: Groups whose all subgroups are special. *Math. Sb.* **31**, 366–372 (1924)
- [14] Schmidt, R.: *Subgroup Lattices of Groups*. Walter de Gruyter & Co., Berlin (1994)
- [15] Tărnăuceanu, M.: Normality degrees of finite groups. *Carpathian J. Math.* **33**(1), 115–126 (2017)
- [16] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.16.0; 2026 (<https://www.gap-system.org>)
- [17] Zassenhaus, H.: *The Theory of Groups*, Translated from the German by Saul Kravetz. Chelsea Publishing Co., New York (1949)

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Received: April 20, 2026.

Revised: August 6, 2026.

Accepted: August 12, 2026.